Solution of the time-dependent Boltzmann equation

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The time-dependent Boltzmann equation, which describes the propagation of radiation from a point source in a random medium, is solved exactly in Fourier space. An explicit expression in real space is given in two and four dimensions. In three dimensions an accurate interpolation formula is found. The average intensity at a large distance r from the source has two peaks, a ballistic peak at time $t=r/c$ and a diffusion peak at $t\simeq r^2/D$ (with c the velocity and D the diffusion coefficient). We find that forward scattering adds a tail to the ballistic peak in two and three dimensions, $\propto(ct-r)^{-1/2}$ and $\propto-\ln(ct-r)$, respectively. Expressions in the literature do not contain this tail. [S1063-651X(97)08907-1]

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I. INTRODUCTION

The spreading of a pulse of particles or radiation through a random medium has attracted considerable attention in several fields of physics, such as astrophysics, optics, acoustics, solid-state physics, and heat conduction [1–3]. In each of these systems it is possible to generate a pulse of energy, consisting of electromagnetic or acoustic waves, or particles. These then propagate through the medium, with a certain intensity $P(\mathbf{r},t)$ at point $\mathbf{r}$ and time t. In the long-time limit it is accurately given by the solution of the diffusion equation

$$P_{\rm diff}(\mathbf{r},t)=\frac{1}{(4\pi Dt)^{d/2}}\exp\left(-\frac{r^2}{4Dt}-ct/l_a\right). \quad (1)$$

Here $r=|\mathbf{r}|$ is the distance to the source, assumed to be isotropic, t is the time after pulse generation, d is the dimension of the system, $D=cl/d$ is the diffusion coefficient, l is the mean free path, for elastic, isotropic scattering, and l_a is the absorption length. We disregard here any interference effects or effects of inelastic scattering.

The diffusion result (1) is very useful, but has certain shortcomings. First of all, it has a nonzero value at every position even though the energy needs some time to propagate from source to the position of interest. Hence for $r>ct$ the correct P should be identically zero. Furthermore, large deviations from the diffusion approximation can be expected at any r, for short times $t<l/c$. More accurate expressions for the probability density as function of position and time have been proposed [4–7], based on the Boltzmann equation (also known as the equation of radiative transfer [8,9]), of which the diffusion equation is the long-time limit.

In this paper we will present exact solutions of the equation of radiative transfer, and compare them with the approximate expression in the literature [4–6]. The solution in one dimension has been given a long time ago by Hemmer [10]

$$P(r,t)=\frac{1}{2}e^{-ct/2l}\left[\delta(r-ct)+\frac{1}{2l}\Theta(ct-r)\times\left(I_0(\sqrt{c^2t^2-r^2}/2l)+ct\frac{I_1(\sqrt{c^2t^2-r^2}/2l)}{\sqrt{c^2t^2-r^2}}\right)\right],\quad d=1 \quad (2)$$

where I_0 and I_1 are Bessel functions, and the step function $\Theta(x)$ is zero for $x<0$ and 1 for $x>0$. We will generalize this solution to higher dimensions, using a path-integral method [11,12]. In two and four dimensions we are able to give explicit expressions. In three dimensions the solution is given in terms of its Fourier transform. Using the results for $d=2$ and 4 we construct an interpolation for $d=3$, which is correct within a few percent, for all r and t, away from the ballistic peak. A qualitative difference with existing results is that in two and three dimensions the ballistic peak at $r=ct$ is accompanied by a tail resulting from single-scattering events. The analytical shape of this tail is $\propto(ct-r)^{-1/2}$ and $\propto-\ln(ct-r)$ for $d=2$ and 3, respectively.

The outline of this paper is as follows. In Sec. II we derive the Fourier transform $P(\mathbf{k},\omega)$ of the intensity $P(\mathbf{r},t)$ for any dimension. In Sec. III we invert the Fourier transform to get the time and position dependent intensity. We give analytical results for $d=2$ and 4, and numerical results plus an interpolation formula for $d=3$. We compare our results with the literature, and discuss the ballistic peak in some detail. We conclude in Sec. IV.

II. CALCULATION OF THE FOURIER-TRANSFORMED INTENSITY

The theory of radiative transfer describes the place $\mathbf{r}$ and time t dependence of the intensity $P(\mathbf{r},t,\hat{\mathbf{s}})$ of radiation, propagating in the direction $\hat{\mathbf{s}}$ (see Fig. 1). It is based on the Boltzmann equation [8,9]

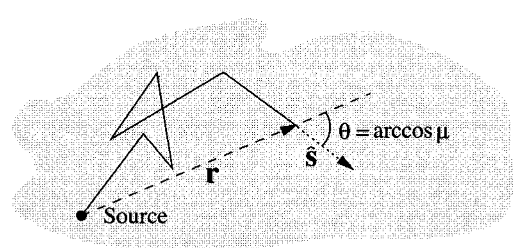

FIG. 1. Schematic drawing of scattering in a random medium. Shown is a single path involving $N=5$ scattering events.

$$\frac{\partial}{c\partial t}P(\mathbf{r},t,\hat{\mathbf{s}})+\hat{\mathbf{s}}\cdot\nabla P(\mathbf{r},t,\hat{\mathbf{s}})$$
$$=-(l^{-1}+l_a^{-1})P(\mathbf{r},t,\hat{\mathbf{s}})+l^{-1}P(\mathbf{r},t)+c^{-1}S(\mathbf{r},t,\hat{\mathbf{s}}), \tag{3a}$$

$$P(\mathbf{r},t)=\int\frac{d\hat{\mathbf{s}}'}{\Omega_d}P(\mathbf{r},t,\hat{\mathbf{s}}'). \tag{3b}$$

Here S is the source term, l is the mean free path for scattering, and l_a is the absorption length. The integration is performed over all directions $\hat{\mathbf{s}}'$ in d dimensions, normalized by the surface area $\Omega_d=2\pi^{d/2}/\Gamma(\frac{1}{2}d)$ of the unit sphere. We have assumed isotropic scattering.

The dependence of the intensity on the absorption is through a $\mathbf{r}$ and $\hat{\mathbf{s}}$ independent factor $\exp(-ct/l_a)$. Without loss of generality we can, therefore, leave the absorption out of our considerations in the following, taking effectively $l_a\to\infty$. We take the isotropic point source

$$S(\mathbf{r},t,\hat{\mathbf{s}})=\delta(\mathbf{r})\delta(t), \tag{4}$$

and seek for a solution to Eq. (3) for $t>0$. (We may set $P\equiv 0$ for $t<0$.) Due to the spherical symmetry, $P(\mathbf{r},t,\hat{\mathbf{s}})$ and $P(\mathbf{r},t)$ only depend on $r=|\mathbf{r}|$, t, and $\mu\equiv\hat{\mathbf{s}}\cdot\mathbf{r}/r$. The Boltzmann equation (3) simplifies to

$$l\left(\frac{\partial}{c\partial t}+\mu\frac{\partial}{\partial r}+\frac{1-\mu^2}{r}\frac{\partial}{\partial\mu}\right)P(r,t,\mu)$$
$$=-P(r,t,\mu)+P(r,t), \tag{5}$$

$$P(r,t)=\int_{-1}^{1}d\mu\rho_d(\mu)P(r,t,\mu). \tag{6}$$

The weight function $\rho_d(\mu)$ is defined by

$$\rho_d(\mu)=\frac{\Gamma(d/2)}{\sqrt{\pi}\Gamma((d-1)/2)}(1-\mu^2)^{(d-3)/2},\quad d>1. \tag{7}$$

In one dimension we find $\rho_1(\mu)=\frac{1}{2}\delta(\mu-1)+\frac{1}{2}\delta(\mu+1)$.

To solve the Boltzmann equation it is useful not to make use of the spherical symmetry initially. We consider separately the contributions to the intensity from $N=0,1,2,\ldots,$ scattering events,

$$P(\mathbf{r},t,\hat{\mathbf{s}})=\sum_{N=0}^{\infty}P_N(\mathbf{r},t,\hat{\mathbf{s}}),\quad P(\mathbf{r},t)=\sum_{N=0}^{\infty}P_N(\mathbf{r},t). \tag{8}$$

Such a decomposition is customary in the theory of random walks [12,13]. It is also at the basis of the path-integral method for the theory of the Boltzmann equation [11]. The partial intensities P_N satisfy

$$\left(\frac{\partial}{c\partial t}+\hat{\mathbf{s}}\cdot\nabla+l^{-1}\right)P_N(\mathbf{r},t,\hat{\mathbf{s}})=l^{-1}P_{N-1}(\mathbf{r},t),\quad N>0, \tag{9a}$$

$$\left(\frac{\partial}{c\partial t}+\hat{\mathbf{s}}\cdot\nabla+l^{-1}\right)P_0(\mathbf{r},t,\hat{\mathbf{s}})=c^{-1}S(\mathbf{r},t). \tag{9b}$$

The differential operators on the left hand side can be integrated, to yield

$$P_N(\mathbf{r},t,\hat{\mathbf{s}})=l^{-1}\int_0^\infty dr_0e^{-r_0/l}P_{N-1}(\mathbf{r}-r_0\hat{\mathbf{s}},t-r_0/c), \tag{10a}$$

$$P_0(\mathbf{r},t,\hat{\mathbf{s}})=c^{-1}\int_0^\infty dr_0e^{-r_0/l}S(\mathbf{r}-r_0\hat{\mathbf{s}},t-r_0/c). \tag{10b}$$

Similarly, we find for the angular average of the intensity

$$P_N(\mathbf{r},t)=\int d\mathbf{r}_0p_0(\mathbf{r}_0)P_{N-1}(\mathbf{r}-\mathbf{r}_0,t-r_0/c), \tag{11a}$$

$$P_0(\mathbf{r},t)=lc^{-1}\int d\mathbf{r}_0p_0(\mathbf{r}_0)S(\mathbf{r}-\mathbf{r}_0,t-r_0/c), \tag{11b}$$

where we defined

$$p_0(\mathbf{r})=\frac{e^{-r/l}}{\Omega_d l r^{d-1}}. \tag{12}$$

Using the source (4) we can give the explicit expression for the ballistic intensities ($N=0$)

$$P_0(\mathbf{r},t,\hat{\mathbf{s}})=e^{-ct/l}\delta(\mathbf{r}-ct\hat{\mathbf{s}})\Theta(t)$$
$$=e^{-ct/l}\frac{\delta(r-ct)\delta(\mu-1^-)}{\Omega_d r^{d-1}\rho_d(\mu)}, \tag{13a}$$

$$P_0(\mathbf{r},t)=\frac{e^{-ct/l}}{\Omega_d r^{d-1}}\delta(r-ct). \tag{13b}$$

The 1^- in the δ function denotes that it is a single-sided δ function having all its weight in the region $\mu\leqslant 1$. The solution of the recursion relations (11) and consequently of Eq. (10) then is

$$P_N(\mathbf{r},t)=l\left[\prod_{i=0}^{N}\int d\mathbf{r}_ip_0(\mathbf{r}_i)\right]\delta\left(ct-\sum_{i=0}^{N}r_i\right)\delta\left(\mathbf{r}-\sum_{i=0}^{N}\mathbf{r}_i\right), \tag{14a}$$

$$P_N(\mathbf{r},t,\hat{\mathbf{s}})=\Omega_d l\left[\prod_{i=0}^{N}\int d\mathbf{r}_ip_0(\mathbf{r}_i)\right]\delta\left(ct-\sum_{i=0}^{N}r_i\right)$$
$$\times\delta\left(\mathbf{r}-\sum_{i=0}^{N}\mathbf{r}_i\right)\delta(\hat{\mathbf{r}}_0-\hat{\mathbf{s}}), \tag{14b}$$

where $\hat{\mathbf{r}}_0=\mathbf{r}_0/|\mathbf{r}_0|$.

Summation over all N of Eq. (10) results in

$$P(\mathbf{r},t,\hat{\mathbf{s}})=P_0(\mathbf{r},t,\hat{\mathbf{s}})+l^{-1}\int_0^\infty dr_0e^{-r_0/l}P(\mathbf{r}-r_0\hat{\mathbf{s}},t-r_0/c), \tag{15}$$

and leads to the spherical analog of the Schwarzschild-Milne equation [8]

$$P(\mathbf{r},t)=P_0(\mathbf{r},t)+\int d\mathbf{r}_0\ \ p(\mathbf{r}_0)P(\mathbf{r}-\mathbf{r}_0,t-r_0/c). \tag{16}$$

At this point, we introduce the Fourier transform

$$P(\mathbf{k},\omega)=\int d\mathbf{r}\int_0^\infty dt\ e^{i(\omega t-\mathbf{k}\cdot\mathbf{r})}P(\mathbf{r},t), \tag{17}$$

which depends only on ω and $k=|\mathbf{k}|$. We first compute the partial intensities $P_N(k,\omega)$, by taking the Fourier transform of $P_N(\mathbf{r},t)$. The expression factorizes into $N+1$ equivalent integrals over $\mathbf{r}_i$, which can be performed. The result is

$$P_N(\mathbf{k},\omega)=c^{-1}l\left(\int_{-1}^{1}\frac{d\mu\rho_d(\mu)}{1-i\omega l/c+ikl\mu}\right)^{N+1} \tag{18a}$$

$$=c^{-1}l\left[\frac{{}_2F_1(\frac{1}{2},1;\frac{1}{2}d;-k^2l^2(1-i\omega l/c)^{-2})}{1-i\omega l/c}\right]^{N+1}, \tag{18b}$$

$$P_N(\mathbf{k},\omega,\hat{\mathbf{s}})=P_{N-1}(k,\omega)\frac{1}{1-i\omega l/c+il\mathbf{k}\cdot\hat{\mathbf{s}}}, \tag{18c}$$

where ${}_2F_1$ is a hypergeometric function. This expression is the frequency and direction dependent analog of the result for a random walk [12].

III. INVERSION OF THE FOURIER TRANSFORM

In the preceding section we have computed the Fourier-transformed intensity for arbitrary dimension d. In this section we invert the Fourier transform, which can be done analytically for $d=2$ and $d=4$, and numerically for $d=3$.

A. Two dimensions

In two dimensions Eq. (18) simplifies to

$$P_N(k,\omega)=c^{-1}l[(1-i\omega l/c)^2+k^2l^2]^{-(N+1)/2}. \tag{19}$$

The ballistic $N=0$ term consists of a δ function in real space and is given in Eq. (13), (where $\Omega_2=2\pi$). After an inverse Fourier transformation with respect to $\mathbf{k}$ we find for $N\geqslant 1$

$$P_N(r,\omega)=\frac{1}{cl^2}\left(\frac{r}{2l(1-i\omega l/c)}\right)^{(N-1)/2}\times\frac{1}{\Gamma((N+1)/2)}K_{(N-1)/2}((1-i\omega l/c)r/l). \tag{20}$$

Using the representation

$$K_\nu(z)=\frac{\sqrt{\pi}(z/2)^\nu}{\Gamma(\nu+\frac{1}{2})}\int_0^\infty d\xi(\sinh\xi)^{2\nu}e^{-z\cosh\xi}, \tag{21}$$

and the substitution $\cosh\xi=ct/r$, one can see that Eq. (20) is the Fourier transform of

$$P_N(r,t)=\frac{e^{-ct/l}}{2\pi l^2}\frac{1}{(N-1)!}\left(\frac{ct}{l}\right)^{N-2}\left(1-\frac{r^2}{c^2t^2}\right)^{(N-2)/2}\times\Theta(ct-r),\ \ N\geqslant 1. \tag{22}$$

Summing over N, and adding the ballistic $N=0$ term from Eq. (13), we find the total intensity

$$P(r,t)=\frac{e^{-ct/l}}{2\pi r}\delta(ct-r)+\frac{1}{2\pi lct}\left(1-\frac{r^2}{c^2t^2}\right)^{-1/2}\times\exp[l^{-1}(\sqrt{c^2t^2-r^2}-ct)]\Theta(ct-r). \tag{23}$$

The diffusion result (1), with $D=cl/2$, is recovered for $t\gg r/c$. It is remarkable that the diffusion approximation does not require that $ct\gg l$, but only that $ct\gg r$. We will see that this is special for two dimensions.

To obtain the angular resolved intensity $P_N(\mathbf{r},t,\hat{\mathbf{s}})$ we perform the integral over r_0 in Eq. (15). The integrand vanishes for $r_0>r_{\max}$, defined by

$$|\mathbf{r}-r_{\max}\hat{\mathbf{s}}|=ct-r_{\max}\Leftrightarrow r_{\max}=\frac{(ct)^2-r^2}{2(ct-r\mu)}. \tag{24}$$

We thus find for the intensity the result ($N\geqslant 1$)

$$P_N(\mathbf{r},t,\mu)=\frac{1}{2\pi l^N(N-2)!}\int_0^{r_{\max}}dr_0e^{-ct/l}[(ct-r_0)^2-(\mathbf{r}-r_0\hat{\mathbf{s}})^2]^{(N-3)/2}$$
$$=\frac{e^{-ct/l}}{2\pi l(N-1)!}\frac{1}{ct-r\mu}\left(\frac{ct}{l}\right)^{N-1}\times\left(1-\frac{r^2}{c^2t^2}\right)^{(N-1)/2}\Theta(ct-r). \tag{25}$$

Summing over N and including the ballistic contribution (13) for $N=0$, we find

$$P(\mathbf{r},t,\hat{\mathbf{s}})=e^{-ct/l}\delta(\mathbf{r}-ct\hat{\mathbf{s}})\Theta(t)+\frac{1}{2\pi l(ct-\mathbf{r}\cdot\hat{\mathbf{s}})}\times\exp[l^{-1}(\sqrt{c^2t^2-r^2}-ct)]\Theta(ct-r). \tag{26}$$

B. Four dimensions

In four dimensions the Fourier-transformed intensity is given by

$$P_N(k,\omega)=2^{N+1}c^{-1}l(\sqrt{(1-i\omega l/c)^2+k^2l^2}+1-i\omega l/c)^{-(N+1)}. \tag{27}$$

To invert the Fourier transform we use

$$\int\frac{d\mathbf{k}}{(2\pi)^4}e^{i\mathbf{k}\cdot\mathbf{r}}f(|\mathbf{k}|)=\frac{1}{4\pi^2r}\int_0^\infty dk\ k^2J_1(kr)f(k), \tag{28}$$

so that

$$P_N(r,t)=\frac{2^{N-1}e^{-ct/l}}{\pi^3 i l^N r^{N+3}}\int_0^\infty dk\ J_1(kr)k^{-2N}$$
$$\times\int_{r/l-i\infty}^{r/l+i\infty}dz\ e^{zct/r}[\sqrt{k^2r^2+z^2}-z]^{N+1}. \quad (29)$$

The integral over z yields

$$i(N+1)(kr)^{N+1}J_{N+1}(kct)\Theta(t)r/ct.$$

After integration over k we find, for $N\geqslant 1$,

$$P_N(r,t)=\frac{1}{\pi^2}e^{-ct/l}\frac{1}{ctl^3}\left(\frac{ct}{l}\right)^{N-3}\times\frac{N+1}{(N-1)!}\left[1-\frac{r^2}{c^2t^2}\right]^{N-1}\Theta(ct-r). \quad (30)$$

Again we sum over all N, and include the ballistic term $N=0$, to find the total intensity

$$P(r,t)=\frac{e^{-ct/l}}{2\pi^2r^3}\delta(r-ct)+\frac{1}{(\pi lct)^2}\left(1-\frac{r^2}{c^2t^2}+\frac{2l}{ct}\right)\times\exp(-r^2/lct)\Theta(ct-r). \quad (31)$$

If both r and l are $\ll ct$ we find the diffusion result (1), with diffusion constant $D=\frac{1}{4}cl$.

In the same way as we did for $d=2$ we can calculate the angular resolved intensity $P(r,t,\mu)$ from Eqs. (15) and (30). We find ($N\geqslant 1$)

$$P(r,t,\mu)=\frac{\pi e^{-ct/l}}{2r^3}\delta(r-ct)\delta(\mu-1^-)(1-\mu^2)^{-1/2}+\frac{\exp(-r^2/lct)}{(\pi lct)^2}\times\frac{(1-y^2)(1+y^2-2\mu y)+2(1-\mu y)l/ct}{(1+y^2-2\mu y)^2}\times\Theta(ct-r), \quad (32)$$

where we have abbreviated $y=r/ct$.

C. Three dimensions

In three dimensions the Fourier-transformed intensity reads

$$P_N(k,\omega)=c^{-1}l\left[\frac{1}{kl}\arctan\left(\frac{kl}{1-i\omega l/c}\right)\right]^{N+1}. \quad (33)$$

The inverse can be evaluated analytically for $N=0$ and $N=1$, but not for arbitrary N. An interpolation between the results (22) and (30) for $d=2$ and 4 suggests the approximation $P_N\propto[1-r^2/(ct)^2]^{3N/4-1}$. The coefficient $\frac{3}{4}$ in the exponent ensures that the diffusion limit is obtained when r and l are $\ll ct$. The definition (14) implies the normalization

$$\int d\mathbf{r}P_N(\mathbf{r},t)=\frac{1}{N!}\left(\frac{ct}{l}\right)^N e^{-ct/l}. \quad (34)$$

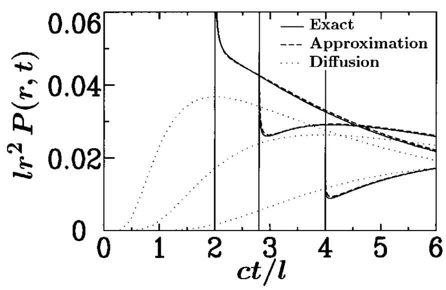

FIG. 2. Angular average of the intensity $P(\mathbf{r},t)$ for three dimensions as a function of time t, at distance $r=2.0l$, $2.8l$, and $4.0l$, from left to right. The solid lines are the exact result (A5), which is very close to the interpolation formula (36) (dashed lines). The dotted lines are the diffusion result (1). The intensity has a minimum for r greater than some r_c.

Taking this normalization into account, we find for $N\geqslant 1$ the approximation

$$P_N(r,t)\simeq\frac{e^{-ct/l}}{\pi l^3}\frac{\Gamma(\frac{3}{4}N+\frac{3}{2})}{\sqrt{\pi}N!\Gamma(\frac{3}{4}N)}\left(\frac{ct}{l}\right)^{N-3}\times\left(1-\frac{r^2}{c^2t^2}\right)^{\frac{3}{4}N-1}\Theta(ct-r). \quad (35)$$

Because of its construction as an interpolation between two exact results, we expect Eq. (35) to be rather accurate.

The total intensity including the ballistic peak, becomes

$$P(r,t)\simeq\frac{e^{-ct/l}}{4\pi r^2}\delta(r-ct)+\frac{(1-r^2/c^2t^2)^{1/8}}{(4\pi lct/3)^{3/2}}e^{-ct/l}\times G\left(\frac{ct}{l}\left[1-\frac{r^2}{c^2t^2}\right]^{3/4}\right)\Theta(ct-r), \quad (36a)$$

$$G(x)=8(3x)^{-3/2}\sum_{N=1}^{\infty}\frac{\Gamma(\frac{3}{4}N+\frac{3}{2})}{\Gamma(\frac{3}{4}N)}\frac{x^N}{N!}\simeq e^x\sqrt{1+2.026/x}. \quad (36b)$$

For $l,r\ll ct$ the diffusion result (1) is regained, with $D=cl/3$.

To check the accuracy of this interpolation we have compared Eq. (36) with a numerical inversion of the Fourier transform (see the Appendix). In Fig. 2 we have plotted the intensity as a function of ct/l for three values of r/l. The dashed curves are the approximation (36). The difference is barely visible on this scale, and is of the order of 2% outside the ballistic peak and its tail.

In Fig. 3 we compare our result with approximations from the literature. Perelman *et al.* [4] have improved upon the diffusion result using a path-integral method, taking the average velocity of the light equal to c, such that the intensity vanishes for $ct<r$. Their result

$$P(r,t)=\frac{\Gamma(3ct/4l+5/2)}{\pi\sqrt{\pi}t^3\Gamma(3ct/4l+1)}\left(1-\frac{r^2}{c^2t^2}\right)^{3ct/4l}\Theta(ct-r), \quad (37)$$

is shown in Fig. 3. It does not contain the ballistic peak and overestimates the diffusion maximum. Another extension of the diffusion result is due to Kaltenbach and Kaschke [5],

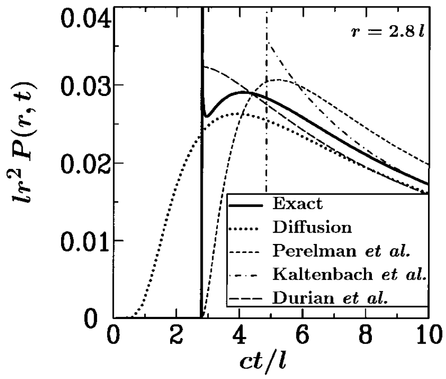

FIG. 3. Average intensity $P(\mathbf{r},t)$ as a function of time t at distance $r=2.8l$. The solid line is the exact result (A5). The dotted line is the diffusion result (1), the short-dashed line is the result (37) from Perelman *et al.* [4], the dashed-dotted line is the result (38) from Kaltenbach and Kaschke [5], and the long-dashed line is the result (40) from Durian and Rudnick [6].

$$P(r,t)=\frac{3\sqrt{3}}{8\pi l^2}\exp(-ct/2l)\Bigg[2\sqrt{3}\delta(ct-r\sqrt{3})l/r + \frac{\Theta(ct-r\sqrt{3})}{\sqrt{c^2t^2-3r^2}}I_1(\sqrt{c^2t^2-3r^2}/2l)\Bigg], \qquad (38)$$

and is also plotted in Fig. 3. The difference with the exact solution is clear. Their method is based on adding to the diffusion equation a second order time derivative, which can be found when deriving the diffusion equation from the Boltzmann equation, but which does not yield the correct ballistic peak. The same was done by Durian and Rudnick [6] but with the prefactor of this second order time derivative chosen such that the ballistic peak is at $t=r/c$. Their result

$$P(r,t)=\frac{e^{-3ct/2l}}{4\pi l^2}\Bigg\{\frac{l^2}{r}\delta'(ct-r)+\left(\frac{3l}{2ct}+\frac{9}{8}\right)\delta(ct-r) \qquad (39)$$

$$+\frac{9}{4l}\Bigg[I_1(3\sqrt{c^2t^2-r^2}/2)+\frac{ctI_2(3\sqrt{c^2t^2-r^2}/2)}{\sqrt{c^2t^2-r^2}}\Bigg]\times\Theta(ct-r)\Bigg\}, \qquad (40)$$

is also plotted in Fig. 3. Again, the difference with the exact solution is clear. Furthermore, this expression introduces the derivative of the δ function at $t=c/r$.

D. Ballistic peak

The main qualitative new feature of our result for the time-dependent angular average of the intensity in two and three dimensions is the tail of the ballistic peak at $t=r/c$ (see Fig. 2). The ballistic peak itself consists of a δ function $P_0\sim\delta(t-r/c)$ due to unscattered radiation. The tail towards larger t is due to radiation which has undergone a single forward scattering event. The shape of the tail is given by P_1, which can be computed analytically for any dimension. The single-scattering intensity is given by Eq. (14) with $N=1$. Since the integration is over only two coordinates, we readily find

$$P_1(r,t,\mu)=\frac{e^{-ct/l}}{l\Omega_d}\frac{1}{ct-r\mu}\frac{1}{(ct-r_{\max})^{d-2}}, \qquad (41)$$

where $r_{\max}$ is given in Eq. (24). Integration over μ with the weight function $\rho_d(\mu)$ given in Eq. (7), yields

$$P_1(r,t)=\frac{2^{d-2}e^{-ct/l}}{\Omega_d l(ct)^{2d-4}}(c^2t^2-r^2)^{(d-3)/2}{}_2F_1\left(\tfrac{1}{2},d-2;\tfrac{1}{2}d;\frac{r^2}{c^2t^2}\right). \qquad (42)$$

For dimensions greater than three the hypergeometric function ${}_2F_1$ has a singularity for $r\to ct$ which is canceled by the factor $(c^2t^2-r^2)^{(d-3)/2}$. The term P_1, therefore, is finite at $r=ct$ and contributes no tail to the ballistic peak for $d>3$. In contrast, for $d\leqslant 3$, the term P_1 has an integrable singularity at $r=ct$, which adds a tail to the ballistic peak. The singularity is logarithmic in three dimensions,

$$P_1(r,t)=\frac{e^{-ct/l}}{4\pi lctr}\ln\frac{ct+r}{ct-r}, \quad d=3 \qquad (43)$$

and algebraic in two dimensions

$$P_1(r,t)=\frac{e^{-ct/l}}{2\pi l}(c^2t^2-r^2)^{-1/2}, \quad d=2. \qquad (44)$$

In one dimension the ballistic peak has no tail, but is enhanced itself by a factor $e^{ct/2l}$ [cf. Eq. (2)].

The tail of the ballistic peak in two and in three dimensions leads to a minimum in the intensity as function of time, provided r is large enough. For this minimum to occur we need $r>[(11+5\sqrt{5})/2]^{1/2}l\simeq 3.330l$ for $d=2$ and $r\gtrsim 2.4l$ for $d=3$. The ballistic peak and its tail are also present in systems with anisotropic scattering. The total intensity in this peak still decreases as $e^{-ct/l}$, l being the mean free path for scattering, whereas the diffusion peak scales with the transport mean free path [9].

IV. CONCLUSION

We have presented exact solutions to the time-dependent Boltzmann equation (or equation of radiative transfer). The method used is based on a summation over the paths, that brings a particle from source to some position $\mathbf{r}$, after N scattering events. This method has been used before, both in connection with the Boltzmann equation [11] and in the theory of random walks [12,13]. However, as far as we know, the exact solution presented here was not known. An important feature of this solution is the tail to the ballistic peak, which has not been noticed in the literature, either in analytical studies [4–6], in experimental results [14], or in numerical simulations based on the Monte Carlo method [13,15,16] (the tail is barely noticeable in the numerical simulations of Ref. [6]). The tail requires a continuum description; it is not present in lattice models [13] for a random walk. Experimentally the observation of the tail is challenging, since the time resolution needed is below the scattering mean free time.

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APPENDIX: NUMERICAL INVERSION OF THE FOURIER TRANSFORM IN THREE DIMENSIONS

In Eq. (33) we gave an analytical expression for $P_N(k,\omega)$ in three dimensions. To find the real space intensity $P(r,t)$ we have to sum over all the number N of scattering events and invert the Fourier transform. In this appendix we show how this can be done numerically. This is not straightforward, because of the singularity at $r=ct$. For notational simplicity we take $l=c=1$ in what follows.

The sum of the contributions for $N\geqslant 4$ is has no singularity and is smooth at $t=r$. It is given by

$$\sum_{N=4}^{\infty} P_N(r,t) = \frac{1}{4\pi^3 r}\int_0^\infty dk\ k^{-3}\sin(kr)\int_{-\infty}^{\infty} d\omega e^{-i\omega t} \times \arctan^5\left(\frac{k}{1-i\omega}\right)\left[k-\arctan\left(\frac{k}{1-i\omega}\right)\right]^{-1}. \tag{A1}$$

The integral over ω can be done by contour integration, closing the contour in the lower half of the complex plane. The contribution from the pole $k=\arctan[k/(1-i\omega)]$ is given by

$$I_{\text{pole}}(k,t)=2\pi\ \exp(tk/\tan k-t)\,\frac{k^2}{\sin^2 k}\,\Theta(\pi/2-k). \tag{A2}$$

To calculate the contribution from the branch cut between $\omega=-i-k$ and $\omega=-i+k$ we parametrize $\omega=-i+\xi k$. We find

$$I_{\text{cut}}(k,t)=\frac{\pi e^{-t}}{4k^2}\int_{-1}^{1}d\xi\left[\cos(kt\xi)\frac{4k^2(5\Lambda^4-10\Lambda^2\pi^2+\pi^4)+(\Lambda^2+\pi^2)^2(3\Lambda^2-\pi^2)}{(4k^2-\Lambda^2-\pi^2)^2+16k^2\Lambda^2}+2\sin(kt\xi)\Lambda \times\frac{2k^2(3\Lambda^2-\pi^2)(\Lambda^2-3\pi^2)+(\Lambda^2-\pi^2)(\Lambda^2+\pi^2)^2}{(4k^2-\Lambda^2-\pi^2)^2+16k^2\Lambda^2}\right], \tag{A3}$$

where we have abbreviated $\Lambda(\xi)=2\ \text{artanh}\ \xi$. Next we calculate the contributions for $N\leqslant 3$. The ballistic term P_0 is already given in Eq. (13) and the single-scattering term P_1 in Eq. (43). For $N=2,3$ we use the same parametrization as above, but interchange the integrals over k and ξ. We find

$$P_2(r,t)=\frac{e^{-t}}{16\pi r}\int_{r/t}^{1}d\xi(3\Lambda^2-\pi^2), \tag{A4a}$$

$$P_3(r,t)=\frac{e^{-t}}{8\pi r}\int_{r/t}^{1}d\xi\Lambda(\Lambda^2-\pi^2)(t\xi-r). \tag{A4b}$$

The total intensity is then given by

$$P(r,t)=\sum_{N=0}^{3}P_N(r,t)+\frac{1}{4\pi^3 r}\times\int_0^\infty dk\ k\ \sin(kr)[I_{\text{pole}}(k,t)+I_{\text{cut}}(k,t)]. \tag{A5}$$

These integrals can be calculated numerically without problems.

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